1 Current Research

I work in the field of low dimensional topology, specifically knot theory. A knot is an embedding of the circle in 3-dimensional space. One can think of this as taking a length of string, tying the strand into a knot, and sealing the ends together. Two knots are equivalent if one can be deformed into the other without cutting the knot or passing it through itself. My research has focused on a generalization of classical knot theory where one studies knots embedded in thickened surfaces. In other words, instead of the ambient space simply being $\mathbb{R}^3$, a knot in a thickened surface is an embedding of the circle in a 2-dimensional surface which is "thickened up" to become a 3-dimensional space. In this context, classical knot theory is equivalent to the case when the thickened surface is a thickened sphere. We define equivalence of these knots analogously, with the reminder that the deformations of the knot must take place inside the thickened surface.

One of the most useful knot invariants is the Alexander module and the associated Alexander polynomial. Scott Carter, Daniel Silver, and Susan Williams [3] define a generalization of the Alexander polynomial to knots in thickened surfaces. In the case where the knot is in the thickened sphere, this new Alexander polynomial is the same as the classical Alexander polynomial. The polynomial is an element of $\mathbb{Z}[t^{\pm 1}, x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_g^{\pm 1}]$ where g is the genus of the surface. Since Carter, Silver, and Williams were able to generalize classical definitions and algorithms, we wanted to generalize a classical result about slice knots.

Two knots are concordant if they cobound a smoothly embedded annulus in $S^3 \times [0, 1]$. Fox and Milnor proved a result which provides an obstruction to two knots being concordant.

Theorem 1.1. *[4] Let $\Delta(K)$ denote the Alexander polynomial of the knot K. If K_0 and K_1 are concordant knots, then $\Delta(K_0) \doteq p(t)p(t^{-1})\Delta(K_1)$, where $p(t) \in \mathbb{Z}[t, t^{-1}]$.*

In the above theorem, the symbol $\doteq$ is meant to indicate that two elements of a ring are equal, up to multiplication by a unit in the ring. One can generalize the idea of concordance to knots in thickened surfaces. Let Σ be a surface of genus $g \geq 1$ and let $K_0, K_1 \subset \Sigma \times [0, 1]$ be knots in a thickened surface. We say that K_0 is concordant to K_1 if the two knots cobound a smoothly embedded annulus in $\Sigma \times [0, 1] \times [0, 1]$. I was able to generalize the Fox-Milnor Theorem.

Theorem 1.2. *[7] Suppose K_0 and K_1 are concordant knots in a thickened surface and let $\Delta(K)$ denote the Alexander polynomial of K as defined by Carter, Silver, and Williams. Then, $\Delta(K_0) \doteq \alpha \cdot \overline{\alpha} \cdot \Delta(K_1)$ where α is an element of $\mathbb{Z}[t^{\pm 1}, x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_g^{\pm 1}]$ and $\bar{\cdot}$ is the involution of the ring sending each variable to its inverse.*

2 Future Research

The study of knots in a thickened surface is related to virtual knot theory. A knot in a thickened surface is a topological realization of a virtual knot diagram, where a virtual crossing represents a part of the knot where the strands wind around the surface. In fact, there is a bijective correspondence between knots in thickened surfaces up to (de)stabilization (addition or reduction of genus) and classes of virtual knots [9]. Carter, Silver, and Williams have had success in using their Alexander polynomial to produce results about virtual knots. Specifically, they were able to produce a lower bound on the virtual genus of a virtual knot, or the smallest genus of a surface supporting a diagram of the virtual knot. I would like to further investigate the relationships between the Alexander polynomial of Carter, Silver, and Williams and virtual knots. Louis Kauffman [6] has introduced the notion of virtual knot concordance which is defined via moves on the knot diagram. Turaev [12] introduced a notion of concordance of knots on thickened surfaces up to (de)stabilization. A result of Carter, Kamada, and Saito [2] can be used to show that Kauffman's definition of virtual knot concordance and Turaev's definition of concordance of knots in thickened surfaces up to (de)stabilization are equivalent. The definition of concordance I have worked with does not allow for (de)stabilization and requires that concordant knots are in surfaces of the same genus. I would like to investigate the connections between all of these definitions. If a clear relationship exists, the Alexander polynomial would then serve as an obstruction to concordance of virtual knots. Since there has been success in generalizing results to knots in a thickened surface, I would like to continue trying to generalize other results. One such result is the fact that if $p(t) \in \mathbb{Z}[t, t^{-1}]$, $p(1) = \pm 1$, and $p(t) = p(t^{-1})$, then $p(t)$ is the Alexander polynomial of some knot.

3 Mentoring Undergraduate Research

In the summer of 2015, I served as a mentor at the Seattle University REU, which was directed by Allison Henrich. The REU focused on recruiting underrepresented and underprivileged students who did not have an extensive mathematics background. Students from community colleges were among the participants. Knot theory is a fantastic area of study to introduce students to a different side of math. These students have for the most part only seen calculus and math taught in a way that pushes formulas, standard procedures, and rote memorization. Knot theory forces students to develop their spatial intuition and reasoning, which is one of the main reasons I was drawn to the topic. Knot theory is very accessible since one often works with diagrams and invariants associated to these diagrams. My goal is to use this accessibility to teach students concepts in higher level math that they might not see in a standard calculus sequence. The students who attended the Seattle University REU were only required to have completed the calculus sequence. Even with a relatively inexperienced group of student, our work resulted in two published papers [1], [8].

There were two groups of knot theory students at the REU. Both groups focused on problems using a new operation on knot diagrams introduced by Ayaka Shimizu, called region crossing change (abbreviated RCC) [11]. The RCC is an example of an unknotting operation, which is an operation on knot diagrams which is capable of turning any diagram into a diagram of the unknot. One group focused on developing a game played on knot diagrams using RCC called the Region Unknotting game [1]. This game is a variation on the Knotting Unknotting game previously developed by Allison Henrich and her students at an REU at Williams College [5]. The game is played on a knot diagram with two players. Players take turns choosing regions of the diagram where they either perform an RCC move or leave the region unchanged. The goal of one player is to have an unknotted diagram at the end of game play, while the other player wants the diagram to be knotted. The other group focused on region dealternating number [8]. An alternating diagram is a knot diagram on which, when traversing the knot in a fixed direction, the crossings encountered alternate between over and under crossings. The alternating number of a diagram is the number of crossings which need to be changed to result in an alternating diagram. One can define an analogous notion of region dealternating number, the number of RCC moves required to produce an alternating diagram. We studied the properties of this knot invariant. Our study of this knot invariant led us to insights which proved useful in solving an open problem posed in [10], resulting in our published work [8].

4 Future Undergraduate Research

There are many interesting variations of knot games that can be studied. We could allow knot games to be played on virtual diagrams. We could allow the use of Reidmeister moves during game play. One particularly interesting suggestion made by Louis Kauffman at a conference was to try to develop a knot cobordism game.

In the summer of 2016, I attended the third Unknot Conference at Denison University, which is an undergraduate knot theory conference. On the last day of the conference, the participants as a group generated a list of 65 open problems that can be tackled by undergraduates. Knot theory is a very active area of research for undergraduates and there are many interesting open questions that are perfect for undergraduates.

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